# Quantum Velocity Space Distributions

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(Z. Naturforsch. **24 a**, 1431 [1969]; received 24 June 1969)

In a recent paper[1], the present author has shown that it is possible to reformulate Schrödinger quantum mechanics in terms of a two fluid phase space or velocity space structure. The distribution which has played a central role in this work, except that the function $\gamma(\tau_1, \tau_2)$ has not before now been introduced into its prescription, has the form

$$m(x, t \mid v_1, v_2) = \varrho^{-1} \frac{(m_0)^3}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \gamma(\tau_1, \tau_2)\, \Gamma(x, t, \tau_1, \tau_2) \cdot e^{-i(\tau_1 v_1 + \tau_2 v_2)}\, d\tau_1\, d\tau_2 \tag{1}$$

where

$$\Gamma(x, t, \tau_1, \tau_2) = \psi^* \left(x - \frac{\hbar \tau}{2}, t\right) \psi \left(x + \frac{\hbar \tau^*}{2}, t\right) \tag{2}$$

$$\tau = \tau_1 + i \tau_2 \tag{3}$$

and

$$\varrho = \psi^*(x, t)\, \psi(x, t)\,. \tag{4}$$

The function $m(x, t \mid v_1, v_2)$ as represented here can be regarded as a mass distribution on the joint fluid velocity space $(v_1, v_2)$. The factor $\varrho^{-1} m_0^3 / 2\pi$ is merely a velocity space normalization function and $m_0$ is the total effective mass of the system. Without the function $\gamma(\tau_1, \tau_2)$, the onus of suitable behaviour for the integrand in the $\tau$ space integrals which arise when averaging rests entirely on the wave function product (2) and cannot be rigorously substantiated. However, the introduction of a function, $\gamma(\tau_1, \tau_2)$, with suitable properties clears up what has been one of the few remaining mathematical defects in this new approach to quantum mechanics.

The function $\gamma(\tau_1, \tau_2)$ will be taken to have properties as follows. Inside the circle,

$$\tau_1^2 + \tau_2^2 = r^2, \quad \gamma(\tau_1, \tau_2) \equiv 1\,. \tag{5}$$

Here $r$ can be a very small but finite real number. Thus we shall have

$$\left.\frac{\partial \gamma}{\partial \tau_1}\right|_{\tau=0} = \left.\frac{\partial \gamma}{\partial \tau_2}\right|_{\tau=0} = 0\,, \quad \left.\frac{\partial^2 \gamma}{\partial \tau_1^2}\right|_{\tau=0} \left.\frac{\partial^2 \gamma}{\partial \tau_2^2}\right|_{\tau=0} = 0\,, \tag{6}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots = 0\,,$$

as far as may be appropriate to the average being calculated. At large distances from the origin, a suitable prescription for $\gamma$ is that for all points in $\tau$ space outside the circle

$$\tau_1^2 + \tau_2^2 = R^2\,,$$
$$\gamma \Gamma = \exp\{-\lambda(\tau_1^2 + \tau_2^2)\}\, f(\tau_1, \tau_2)\,. \tag{7}$$

The function $\Gamma$ in (7) is the given wave function combination (2), $\lambda$ is a real constant in $\tau$ space and $R$ can be a large real number. $f(\tau_1, \tau_2)$ is some bounded complex function of at least the two variables $\tau_1$ and $\tau_2$, though not a function of the complex variable, $\tau_1 + i\tau_2$. Thus we shall have

$$\gamma \Gamma|_{|\tau|=0} = 0\,,$$
$$\left.\frac{\partial \gamma \Gamma}{\partial \tau_1}\right|_{|\tau|=0} = \left.\frac{\partial \gamma \Gamma}{\partial \tau_2}\right|_{|\tau|=0} = 0\,,$$
$$\left.\frac{\partial^2 \gamma \Gamma}{\partial \tau_1^2}\right|_{|\tau|=0} = \left.\frac{\partial^2 \gamma \Gamma}{\partial \tau_2^2}\right|_{|\tau|=0} = 0\,, \tag{8}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots = 0\,,$$

again as far as may be appropriate to the work in hand. It will be seen that given the combination $\Gamma(x, t \mid \tau_1, \tau_2)$, it is clear that such a ‚plateau' function $\gamma(\tau_1, \tau_2)$, satisfying the conditions (5) and (7) must always exist.

To see how this scheme works consider the average of $v_1^2$. After making use of the properties (8), one finds that

$$\frac{1}{m_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_1^2\, m(x, t \mid v_1, v_2)\, dv_1\, dv_2 \tag{9}$$
$$= -\frac{\varrho^{-1}}{m_0} \left[\frac{\partial^2 \gamma}{\partial \tau_1^2} \Gamma + 2 \frac{\partial \gamma}{\partial \tau_1} \frac{\partial \Gamma}{\partial \tau_1} + \gamma \frac{\partial^2 \Gamma}{\partial \tau_1^2}\right]_{\tau=0}$$
$$= -\frac{\varrho^{-1}}{m_0} \left[\frac{\partial^2 \Gamma}{\tau_1^2}\right]_{\tau=0}, \tag{10}$$

after using (5) and (6). Expression (10) is one of the results used in reference [1] where $\gamma \equiv 1$ everywhere and boundary terms at infinity in $\tau$ space were dropped by the optimistic use of Eq. (18). The introduction of $\gamma(\tau_1, \tau_2)$ improves the mathematics of this new formalism greatly and makes no significant difference to the physical conclusions. It is perhaps useful to regard $\gamma(\tau_1, \tau_2)$ as representing a clumping of contributions from the various possible $\Gamma(x, t \mid \tau_1, \tau_2)$ near the origin in $\tau$ space.

[1] J. G. Gilson, Z. Naturforsch. **24 a**, 198 [1969].